

Lecture 15 (March 28, 2016)

Read: contraction method for nonlinear systems: a brief introduction and some open problems (Aminzare, Sontag, IEE CDC 2014)

contraction theory

Global stability is a central research topic in dynamical systems theory. Stability properties are typically defined in terms of attraction to an invariant set (eq. pt or limit cycle) often coupled with a Lyapunov stability requirement.

An stronger requirement than attraction to a pre-specified target set is to ask that any two trajectories should converge to each other, exponentially and with no overshoot.

In contrast to Lyapunov method, contraction methods, do not require the prior knowledge of attractors.

Proper tool to check contractivity for nonlinear systems is provided by logarithmic norm (matrix measure).

History: The idea is a classical one: Lewis 1940, Dahlquist 1958, Demidovic & Yoshizawa 1960.

In control theory, Lohmiller & Slotine 1998.

Some applications: stability, observer problems, consensus problems in complex networks.

Recall: Let $\|\cdot\|$ be an induced matrix norm on $\mathbb{R}^{n \times n}$. Then the corresponding matrix measure (logarithmic norm) is the function $M: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ defined by

$$M(A) := \lim_{h \rightarrow 0^+} \frac{\|I + hA\| - 1}{h}$$

The matrix measure can be thought of as the directional derivative of the induced norm $\|\cdot\|$, in the direction of A & evaluated at identity matrix I

- Matrix measure induced by p -norms denoted by M_p and for common p -norms, $p=1, 2, \infty$, can be written as follows:

$$M_1(A) = \max_{j=1 \dots n} |a_{jj}| + \sum_{i=1, i \neq j}^n |a_{ij}| \quad M_\infty(A) = \max_{i=1 \dots n} |a_{ii}| + \sum_{j=1, j \neq i}^n |a_{ij}|$$

$$M_2(A) = \lambda_{\max} \frac{A + A^T}{2} \quad \text{Example. } A = \begin{pmatrix} -4 & 0 \\ -2 & -3 \end{pmatrix}$$

Some properties of M :

Let $A, B \in \mathbb{R}^{nm}$ be an arbitrary matrix.

① $\lambda_{\max}(A) \leq M(A) \leq \|A\|$ ← matrix op. norm induced by $\|\cdot\|$.

② Note that unlike norm, logarithmic norms could be negative.

$$A = \begin{pmatrix} -3 & -1 \\ 2 & -4 \end{pmatrix} \quad M_1(A) = -1 < 0 < \|A\|_1 = 5$$

③ Unlike norms, one cannot compare logarithmic norms:

$$A = \begin{pmatrix} -3 & -1 \\ 2 & -4 \end{pmatrix} \quad M_2(A) \approx -2.8 < M_1(A) = -1 < 0$$

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad M_2(B) = 2.2 > M_1(B) = 2 > 0$$

④ Subadditivity property

$$\text{i)} M(A+B) \leq M(A) + M(B) \quad \text{ii)} M(\alpha A) = \alpha M(A) \quad \alpha \geq 0$$

In 1965, Coppel showed that the matrix measure can be used to bound solutions of a linear differential equation $\dot{x} = A(t)x$:

Theorem 1. If $A(t)$ is a continuous matrix function defined for $t \geq t_0$, then for any solution of $\dot{x} = A(t)x$, and any $t \geq t_0$,

$$\|x(t_0)\| \exp\left(-\int_{t_0}^t M(-A(s))ds\right) \leq \|x(t)\| \leq \|x(t_0)\| \exp\left(\int_{t_0}^t M(A(s))ds\right)$$

In 1970, Martin introduced a generalization of the logarithmic norm and proved Theorem 1 for nonlinear differential equations:

Def. Let $\|\cdot\|$ be a norm on \mathbb{R}^n and $f: V \rightarrow \mathbb{R}^n$ be a function where $V \subset \mathbb{R}^n$. The least upper bound (lub) Lipschitz constant of f induced by norm $\|\cdot\|$, on V , is defined by

$$L[f] = \sup_{x \neq y \in V} \frac{\|f(x) - f(y)\|}{\|x - y\|} \quad (\text{generalization of matrix of norm})$$

Note that $L[f] < \infty$ iff f is Lipschitz on V .

Def. Let $\|\cdot\|$ be a given norm on \mathbb{R}^n and $f: V \rightarrow \mathbb{R}^n$, $V \subset \mathbb{R}^n$, be a Lipschitz function. The least upper bound (lub) logarithmic Lipschitz constant of f induced by $\|\cdot\|$, on V , is defined by

$$\begin{aligned} \mu[f] &= \lim_{h \rightarrow 0^+} \frac{1}{h} \{ L[I + hf] - 1 \} \\ &= \lim_{h \rightarrow 0^+} \sup_{x \neq y} \frac{1}{h} \left\{ \frac{\|x - y + h(f(x) - f(y))\|}{\|x - y\|} - 1 \right\} \end{aligned}$$

Theorem 2. Consider $\dot{x} = f(t, x)$ and suppose that f is a Lipschitz function of x and a continuous function of t , $t \geq t_0$. Then, for any solution $x(t)$ and any $t \geq t_0$:

$$\|x(t_0)\| \exp\left(-\int_{t_0}^t \mu(-f_s) ds\right) \leq \|x(t)\| \leq \|x(t_0)\| \exp\left(\int_{t_0}^t \mu(f_s) ds\right)$$

$$f_s(\cdot) = f(s, \cdot)$$

Remark. $\mu[f]$ can be defined for any operator f defined on any normed space $(X, \|\cdot\|_X)$. In this course, since we are interested in vector fields defined on \mathbb{R}^n , we let $X = \mathbb{R}^n$.

Lemma 1. Let $G: [0, \infty) \times V \rightarrow \mathbb{R}^n$ be C^1 with respect to the second argument, where $V \subseteq \mathbb{R}^n$. Suppose $u & v: [0, \infty) \rightarrow V$ satisfy

$$(\dot{u} - \dot{v})(t) = G(t, u(t)) - G(t, v(t))$$

Let $c = \sup_{(t,x)} \mu(J_G(t,x))$, where μ is the matrix measure induced by a given norm $\|\cdot\|$. Then for all $t \geq 0$,

$$\|u(t) - v(t)\| \leq e^{ct} \|u(0) - v(0)\| \quad (*)$$

Note that $(*)$ estimates rate of contraction ($c < 0$) and expansion ($c > 0$).

Proof. Let $z(t) = u(t) - v(t)$. we have that $\dot{z}(t) = A(t) z(t)$ where

$$A(t) = \int_0^1 \frac{\partial f}{\partial x} (su(s) + (1-s)v(s)) ds. \text{ Now by subadditivity of matrix}$$

measure, which by continuity extends to integrals, we have :

$$\mu(A(t)) \leq \sup_x \mu(J_f(t,x)) \Rightarrow \sup_t \mu(A(t)) < c$$

Applying Coppel's inequality, gives the result.

Def. Given a norm $\|\cdot\|$, the nonlinear, time dependent system $\dot{x} = f(t,x)$ or the vector field f , is said to be infinitesimally contracting with respect to this norm on a set $V \subseteq \mathbb{R}^n$ if there exists some norm in V with associated matrix measure μ st. for some constant $c > 0$ (the contraction rate) it holds that: $\mu(J_f(t,x)) \leq -c \quad \forall x \in V, \forall t \geq 0$

The key result is that infinitesimal contractivity implies global contractivity:

Theorem 3. Suppose that V is a convex subset of \mathbb{R}^n and $\dot{x} = f(t,x)$ is infinitesimal contracting with respect to a norm $\|\cdot\|$, with contraction rate c . Then for every two solutions $x(t)$ and $y(t)$ of $\dot{x} = f(t,x)$, that remain in V , it holds that: $\|x(t) - y(t)\| \leq e^{-ct} \|x(0) - y(0)\| \quad (*)$

Proof. since $\dot{x} - \dot{y} = f(t, x) - f(t, y)$ and $\sup_{(t, x)} M(\mathcal{J}f(t, x)) = -c$, by Lemma 1, $(*)$ can be obtained.

The significance of Thm 3 is that it is true for any norm. Different norms are appropriate for different problems, just as different Lyapunov functions have to be carefully chosen when analyzing a nonlinear system. The choice of norms is a key step in the application of contraction techniques.

Example. Consider a standard observer configuration:

$$\dot{x} = f(x, u) \quad \dot{z} = f(z, u) + K(hz - h(x))$$

where h is an output function and K is an observer gain matrix. Let

$$G_t(y) := f(y, u(t)) + K h(y)$$

Then, $\dot{z} - \dot{x} = G_t(z) - G_t(x)$, and thus if G_t has a contractivity property it follows that $z - x$ converges to 0 exponentially, by Lemma 1. (Thm 3 does not apply, since z & x solve different equations.)

This recovers the standard Luenberger observer construction for linear time invariant systems.

Corollary 1. Under the assumptions of Theorem 3, the following statements holds.

1) If A is a non-empty, forward invariant set for the dynamics, then every solution must approach A . Indeed, take any trajectory $x(t)$ & a trajectory $y(t)$ with $y(0) \in A$. Then, as $t \rightarrow \infty$,

$$\text{dist}(x(t), A) \leq \|x(t) - y(t)\| \leq e^{-ct} \|x(0) - y(0)\| \rightarrow 0$$

2) If an eq. pt. exists, then it must be unique & g.a.s.

Relation to Lyapunov stability

Lemma 2. Suppose that P is a positive definite matrix and A is an arbitrary matrix.

- ① If $M_2(PAP^{-1}) = \mu$, then $QA + A^T Q \leq 2\mu Q$, where $Q = P^2$.
- ② If $QA + A^T Q \leq 2\mu Q$, for some $\mu > 0$, then $\exists P > 0$ s.t. $P^2 = Q$ and $M_2(PAP^{-1}) \leq \mu$.

Remark. Lemma 2 implies that, for linear time-invariant systems $\dot{x} = Ax$, contractivity with respect to some weighted L^2 norm, i.e., $M_2(PAP^{-1}) < 0$, is equivalent to A being Hurwitz.

contractivity $\xrightarrow{\text{Thm 3}}$ stability $\Rightarrow A$: Hurwitz

A Hurwitz $\Rightarrow V(x) = x^T Q x$ is a Lyapunov function for some $Q > 0$

that satisfies Lyapunov equation $QA + A^T Q = -B$, $B > 0$. Hence, $\exists \lambda > 0$ s.t. $QA + A^T Q = -B \leq -\lambda_{\min}^B I \leq -\lambda_{\min}^B / \lambda_{\max}^Q Q$

$$\lambda_{\min}^B I \leq B \quad Q \leq \lambda_{\max}^Q I$$

Lemma 2, ②

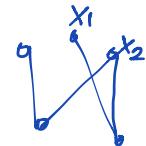
$$\xrightarrow{\quad} M_2(PAP^{-1}) \leq -\frac{1}{2} \frac{\lambda_{\min}^B}{\lambda_{\max}^Q} < 0$$

\therefore contractivity

Diffusive interconnection of identical nonlinear dynamical systems

N identical compartments x_1, \dots, x_N

connected through an arbitrary connected graph



$$\left\{ \begin{array}{l} \dot{x}_i = f(t_i x_i) + D \sum_{j \in N_i} (x_j - x_i) \\ \vdots \\ \dot{x}_N = f(t_N x_N) + D \sum_{j \in N_N} (x_j - x_i) \end{array} \right. \quad \begin{array}{l} x_i \in \mathbb{R}^n, f = (f_1, \dots, f_n)^T \\ N_i: \text{set of all } x_j \text{'s connected to } x_i. \\ D: \text{diagonal, positive matrix} \end{array}$$

Synchronization: $(x_i - x_j) \xrightarrow{t \rightarrow \infty} 0$

Theorem 4. Consider (4). Let $c = \sup_{(t,x)} M [J_f(t,x)]$ where M is induced by a Q -weighted p -norm $\|x\|_{P,Q} = \|Qx\|_p$ for some diagonal, positive Q and $1 \leq p \leq \infty$. Then for any two solutions x & y of (4),

$$\|x(t) - y(t)\|_{P,Q} \leq e^{ct} \|x(0) - y(0)\| \quad (\#)$$

Moreover, if $c < 0$, (4) synchronizes.

Proof of synchronization:

Let $X(t)$ be a solution of (4) with $X(0) = (x_1^0, \dots, x_N^0)^T$.

Let $y(t)$ be the unique solution of $\dot{z} = f(t,z)$ with $z(0) = \frac{\sum x_i^0}{N}$ then $\bar{X}(t) = (y(t), \dots, y(t))^T$ is a solution of (4).

By (#) $\left\| \begin{pmatrix} x_1(t) - y(t) \\ \vdots \\ x_N(t) - y(t) \end{pmatrix} \right\| \xrightarrow[t \rightarrow \infty]{} 0$. Therefore, $x_i - y(t) \rightarrow 0$ as $t \rightarrow \infty$.

More interesting and challenging problem is to provide a condition that links the vector field f , the diffusion matrix D and the graph structure and guarantees synchronization.

Example Suppose that N identical compartments are connected through a complete graph: $\dot{x}_i = f(t, x_i) + D \sum_{j \neq i} (x_j - x_i)$

and let $c = \sup_{(t,x)} M(J_f(t,x) - ND) < 0$ where M is induced by an arbitrary norm on R^n . \nwarrow second eigenvalue of Laplacian matrix

Then the system synchronizes.

Open problems: The problem is understood for L_2 norms & arbitrary graph. The problem is studied for special graph (complete, line, star and their cartesian product) and for Q -weighted p -norms (non- L_2 norms) But the problem is open for arbitrary graphs.